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J. Math. Anal. Appl. 362 (2010) 350–354

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

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Nonwandering set of points of skew-product maps with base having closed set of periodic points ☆

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ARTICLE INFO

Article history:
Received 17 November 2008
Available online 13 August 2009
Submitted by C.E. Wayne

Keywords:
Discrete dynamical system
Skew-product map
Nonwandering point

ABSTRACT

The aim of the present paper is to study the structure of the nonwandering set of points $\Omega(\cdot)$ for the skew-product maps $\mathcal{C}^*_\triangle(\mathbb{I}^2)$ of the unit square $\mathbb{I}^2 = [0,1] \times [0,1]$, $(x,y) \to (f(x), g(x,y))$, with base f having closed set of periodic points. For every $F \in \mathcal{C}^*_\triangle(\mathbb{I}^2)$ and every point (x,y) with x periodic of period p_x by f and y not chain recurrent of $F^{p_x}|_{I_x}$, where $I_x = \{x\} \times \mathbb{I}$, we prove that $(x,y) \notin \Omega(F)$. On the other hand we construct a map $F_0 \in \mathcal{C}^*_\triangle(\mathbb{I}^2)$ with an isolated fixed point x_0 of f and $y_0 \notin \Omega(F|_{I_{x_0}})$ such that $(x_0, y_0) \in \Omega(F_0)$.

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1. Motivation of the research and notation

Discrete dynamical system that are induced by the iteration of a continuous self-map ψ defined on a space $\mathbb{X}$ (which is usually compact and metric) have received a lot of attention in the literature (see for instance [8] or [11]). One reason is that they provide good examples of problems coming from the theory of Topological Dynamics and on the other hand they model many phenomena from biology, physics, chemistry, engineering and social sciences (see for example, [10], [24] or [25]).

In many cases the models are described by two-dimensional systems, for that reason our frame of working will be the well-known class of discrete dynamical systems induced by *skew-product or triangular maps* defined on the unit square $\mathbb{I}^2 = [0,1] \times [0,1]$, i.e., continuous transformations from $\mathbb{I}^2$ into itself of the form $(x,y) \to (f(x), g(x,y))$. The maps f and g are respectively called the *base* and the *fiber* map of F. Obviously, for every $x \in \mathbb{I}$, the maps g_x defined by $g_x(y) := g(x,y)$ form a system of one-dimensional maps depending continuously on x. For having more information on these type of systems, see for instance [1], [4], [5], [19] or [20].

By $\mathcal{C}_\triangle(\mathbb{I}^2)$ we denote the class of skew-product maps on the unit square. Let F be an element of $\mathcal{C}_\triangle(\mathbb{I}^2)$: for every $x \in \mathbb{I}^2$ and every integer $n \geqslant 1$, we define $F^n(x) = F(F^{n-1}(x))$ and F^0 as the identity map on $\mathbb{I}^2$.

A point $x \in \mathbb{I}^2$ is *periodic* by F if there exists a positive integer n such that $F^n(x) = x$. The smallest of the values n satisfying the previous condition is called the *period* of x. By $\mathrm{P}(F)$ we denote the set of all periodic points by F and by $\mathrm{Per}(F)$ we denote the set of all periods of the points of $\mathrm{P}(F)$. The set of periodic points of period 1 is denoted by $\mathrm{Fix}(F)$. For an $x \in \mathbb{I}^2$, we define the *ω-limit set* $\omega_F(x)$ of x by F as the set of all $y \in \mathbb{I}^2$ such that there exists a sequence of positive integers $\{n_k\}_{k=0}^\infty$ holding $F^{n_k}(x) \to y$, where $k \to \infty$. Let $\mathrm{Rec}(F)$ be the set of *recurrent* points of F, i.e., the set of all $x \in \mathbb{I}^2$

☆ This work has been partially supported by MCI (Ministerio de Ciencia e Innovación) and FEDER (Fondo Europeo Desarrollo Regional), grant number MTM2008-03679/MTM, Fundación Séneca de la Región de Murcia, grant number 08667/PI/08 and JCCM (Junta de Comunidades de Castilla-La Mancha), grant number PEII09-0220-0222.
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doi:10.1016/j.jmaa.2009.08.007

such that x is an accumulation point of $(F^n(x))_{n=0}^{\infty}$. A closed invariant set $M \subseteq \mathbb{I}^2$ (i.e., $F(M) \subseteq M$) is called *minimal* by F if it is nonempty and it does not contain proper closed invariant subsets. By $\mathrm{UR}(F)$ we denote the set of *uniformly recurrent* points of F, i.e., all recurrent points with minimal ω-limit sets. A point $x \in \mathbb{I}^2$ is a *nonwandering point* of F provided that for any neighborhood $\mathcal{U}$ of x there exists a positive integer m such that $F^m(\mathcal{U}) \cap \mathcal{U} \neq \emptyset$. The set of nonwandering points of F is denoted by $\Omega(F)$. A point $x \in \mathbb{I}^2$ is a *chain recurrent* point of F if for any $\varepsilon > 0$ there exist points $x_1 = x, x_2, \ldots, x_{n-1}, x_n = x$ (n depends on ε) such that $d(F(x_i), x_{i+1}) < \varepsilon$ for $1 \leqslant i < n$. The set of chain recurrent points of F is denoted by $\mathrm{CR}(F)$. A pair of points $\{x, y\} \subset \mathbb{I}^2$ is said to be a *Li–Yorke* pair of F, if simultaneously is held $\liminf_{n\to\infty} d(F^n(x), F^n(y)) = 0$ and $\limsup_{n\to\infty} d(F^n(x), F^n(y)) > 0$. Given a subset $A \subseteq \mathbb{I}^2$, we say that $F|_A$ is *chaotic* (in the sense of Li–Yorke, see [23]) if A contains a Li–Yorke pair of F.

To detect the presence of simple dynamics in a given discrete system is an important problem in mathematics. The Bowen's definition (see [7]) of the notion of *topological entropy* is a good tool for reaching this aim. In this setting, if the system has zero topological entropy the dynamical behaviour can be understood as simple. On the contrary if the entropy is positive a complex dynamics could appear. Therefore, a natural problem arrives: to find topological characterizations of the notion of zero topological entropy. For interval systems, i.e., discrete systems of the form $(\mathbb{I}, f)$, where f is a continuous self-map of $\mathbb{I}$, there exists a long list of equivalent properties to $(P1)$: f has zero entropy ($h(f) = 0$) (see [27]). Some of the most representative of such properties are the following:

$(P2)$: the topological entropy of $f_{|\mathrm{Rec}(f)}$ is 0 ($h(f_{|\mathrm{Rec}(f)}) = 0$),
$(P3)$: $f_{|\mathrm{Rec}(f)}$ is non-chaotic,
$(P4)$: every recurrent point of f is uniformly recurrent ($\mathrm{Rec}(f) = \mathrm{UR}(f)$),
$(P5)$: the period of every periodic point is power of two.

The equivalence between $(P1)$–$(P5)$ for the interval case establishes an useful procedure for detecting dynamical simplicity.

In 1989, A.N. Sharkovskiĭ and S.F. Kolyada (see [26]) formulated the problem of studying the relations between the properties $(P1)$–$(P5)$ on the setting of skew-product maps of the unit square. It is well known that they are not mutually equivalent (see [3], [13], [17], [19] or [21]). Moreover, even under some additional assumptions on the skew-product map F, the equivalence is not reached. In [18], Z. Kočan proved that in the case of skew-product maps non-decreasing on the *fibres* (i.e., on sets of the form $I_x = \{x\} \times \mathbb{I}$, $x \in \mathbb{I}$) conditions $(P1)$, $(P2)$ and $(P5)$ are equivalent, $(P3)$ implies $(P4)$ and $(P4)$ implies $(P1)$. However (see [18, cf. Lemma 4.2]) there exists an example of a skew-product map non-decreasing on the fibres holding $(P2)$ but nor $(P3)$ neither $(P4)$ (this example is based on the ideas from [14]). The implication from $(P4)$ to $(P3)$ has been recently disproved by J. Chudziak et al. [9] using a Floyd–Auslander minimal system and taking its appropriate continuous extension to a skew-product map of the square non-decreasing on the fibres.

J. Chudziak et al. [15] considered the problem of the equivalence of $(P1)$–$(P5)$ in the class $\mathcal{C}^*_\Delta(\mathbb{I}^2)$ of all skew-product maps with base map having closed set of periodic points. Under such assumption was proved that conditions $(P1)$–$(P5)$ are mutually equivalent. Therefore, it is enough motivated the importance of studing the dynamics of the elements of the class $\mathcal{C}^*_\Delta(\mathbb{I}^2)$. Note that for every $F \in \mathcal{C}^*_\Delta(\mathbb{I}^2)$ its base map f has zero topological entropy (see [8]). We will focused our attention on the structure of the set of nonwandering points.

2. The class $\mathcal{C}^*_\Delta(\mathbb{I}^2)$ and statement of the main results

The class $\mathcal{C}^*_\Delta(\mathbb{I}^2)$ which works for solving the problem stated by Sharkovskiĭ and Kolyada, was studied by L. Efremova [12] in the nineties of the last century. One of the main results in the dynamics of these maps is the following assertion.

Result *(Efremova). Let $F \in \mathcal{C}^*_\Delta(\mathbb{I}^2)$ with base map f. Then*

$$\Omega(F) = \overline{\bigcup_{x \in \mathrm{P}(f)} \{x\} \times \Omega\big(F^{p_x}|_{I_x}\big)}, \tag{1}$$

where p_x is the period of x.

This result had some important implications, one of them, quoted several times in the literature (see for instance the seminal paper [19] or the monograph [27]), was the equivalence between the following two properties:

- $\mathrm{P}(F)$ is closed,
- $\mathrm{P}(F) = \Omega(F)$.

Recently, J.L.G. Guirao et al. [16] showed that the previous equivalence does not hold disproving the Efremova's result. This shows that the dynamics of maps of $\mathcal{C}^*_\Delta(\mathbb{I}^2)$ is still far from being understood. Our aim in this paper is to study the nonwandering set of points for maps on $\mathcal{C}^*_\Delta(\mathbb{I}^2)$. The statement of our main results is the following.

Theorem A. *Let $F \in \mathcal{C}^*_\Delta(\mathbb{I}^2)$ with base map f. If $x \in \mathrm{P}(f)$ and $y \notin \mathrm{CR}(F^{p_x}|_{I_x})$ then $(x, y) \notin \Omega(F)$.*

Theorem. *There exists $F_0 \in \mathcal{C}^*_\Delta(\mathbb{I}^2)$ with the base map f such that $x_0 \in \mathrm{Fix}(f)$ is isolated in $\mathrm{P}(F)$, $y_0 \notin \Omega(F|_{I_{x_0}})$ and (x_0, y_0) is a nonwandering point of the map F_0.*

In the last section we stated some conclusions and an open problem on $\mathcal{C}^*_\Delta(\mathbb{I}^2)$ related with the nonwandering set of points.

3. Proof of Theorem A

For obtaining the proof we shall use two auxiliary results from [22].

Lemma 1. *Let f be a continuous interval map with closed set of periodic points. Then, for any $x \in \mathrm{Fix}(f)$, there is an arbitrarily small open neighborhood $\mathcal{U}$ of x satisfying*:

$$\text{if } s \in \mathcal{U} \text{ but } f(s) \notin \mathcal{U} \text{ then } f^n(s) \notin \mathcal{U} \text{ for all } n > 0. \tag{2}$$

Lemma 2. *$F \in \mathcal{C}^*_\Delta(\mathbb{I}^2)$ and let $(x, y) \in \Omega(F)$. Then x is periodic of the base and if p_x denotes its period then*

$$\Omega\big(F^p\big) \cap \big(\{x\} \times \mathbb{I}\big) = \Omega(F) \cap \big(\{x\} \times \mathbb{I}\big). \tag{3}$$

Proof of Theorem A. Let the assumptions be fulfilled. We show that $z = (x, y)$ is wandering, i.e., there exists an open neighborhood $W \subseteq \mathbb{I}^2$ of z of the form $\mathcal{W} = \mathcal{U} \times \mathcal{V}$, where $\mathcal{U}, \mathcal{V} \subseteq \mathbb{I}$, such that

$$\mathcal{W} \cap F^m(\mathcal{W}) = \emptyset \quad \text{for any } m > 0. \tag{4}$$

By Lemma 2, we may assume that x is a fixed point. For simplicity let $g = F|_{I_x}$. It is known that $y \notin \mathrm{CR}(g)$ if and only if there exists an open set $\mathcal{U} \subseteq \mathbb{I}$ with $g(\overline{U}) \subseteq \mathcal{U}$ such that $y \notin \overline{\mathcal{U}}$ and $g(y) \in \mathcal{U}$ (see [8]).

Let $\|\cdot\|$ be the uniform norm on the space of continuous self-maps on the interval $\mathbb{I}$. Then there is an $\varepsilon > 0$ such that, for any continuous $h : \mathbb{I} \to \mathbb{I}$ with $\|g - h\| < \varepsilon$, $h(y) \in \mathcal{U}$ and $h(\overline{\mathcal{U}}) \subseteq \mathcal{U}$. Since $y \notin \overline{\mathcal{U}}$ there is an open neighborhood $\mathcal{V} \subseteq I_x$ of y such that

$$dist\left(\mathcal{V}, \bigcup_{i>0} h^i(\mathcal{V})\right) > 0 \quad \text{for any } h : \mathbb{I} \to \mathbb{I} \text{ continuous, } \|g - h\| < \varepsilon.$$

Moreover, by Lemma 1 there exists an open neighborhood $\mathcal{K}$ of x, $\mathcal{K} \subseteq (x - \varepsilon, x + \varepsilon)$, which satisfies (2). Clearly, $\mathcal{W} = \mathcal{K} \times \mathcal{V}$ satisfies (4) and hence $z \notin \Omega(F)$ ending the proof. □

4. Proof of Theorem B

Let consider f be a continuous interval map with $\mathrm{P}(f)$ closed and for which there exists an attracting fixed point x_0 isolated in $\mathrm{P}(f)$.

Let $g : \mathbb{I} \to \mathbb{I}$ be a piecewise continuous linear map defined by points $g(0) = \frac{7}{8}$, $g(\frac{1}{8}) = \frac{5}{8}$, $g(\frac{1}{4}) = \frac{7}{8}$, $g(\frac{3}{8}) = \frac{1}{8}$, $g(\frac{5}{8}) = \frac{5}{8}$, $g(\frac{3}{4}) = \frac{7}{8}$, $g(1) = 1$ (see Fig. 1). Obviously,

$$y_0 := \frac{1}{8} \in \mathrm{CR}(g) \setminus \Omega(g). \tag{5}$$

However, a very small change of $g(y)$ causes that y_0 becomes a nonwandering point. More precisely, for any sufficiently small $\varepsilon > 0$ ($\frac{1}{16} > \varepsilon$), $y_0 \in \Omega(g_\varepsilon)$ whenever g_ε is a piecewise continuous linear interval map defined by $g_\varepsilon(0) = \frac{7}{8}$, $g_\varepsilon(\frac{1}{8} - \varepsilon) = \frac{5}{8} - \varepsilon$ and

$$g(y) = g_\varepsilon(y) \quad \text{for any } y \in \left[\frac{1}{8}, 1\right].$$

Clearly, for any left-side open neighborhood $\mathcal{W}$ of the fixed point $a = \frac{5}{8}$ there exists $k \in \mathbb{N}$ such that

$$y_0 \in g^{k-1}(\mathcal{W}). \tag{6}$$

Now we define the skew-product map F_0. Let the map f mentioned above be the base map. Since the fixed point x_0 is attracting there exists an open neighborhood $\mathcal{U}$ of x_0 with $diam(\mathcal{U}) < \frac{1}{16}$. Define, for any $x \in \mathcal{U}$, $g_x(y) = g_\varepsilon(y)$ where $\varepsilon = |x - x_0|$. Clearly, this map $F_0 : \mathcal{U} \times \mathbb{I} \to \mathcal{U} \times \mathbb{I}$ can be extended continuously to the whole square $\mathbb{I}^2$.

We are going to show that $z_0 \in \Omega(F_0) \setminus \bigcup_{x \in \mathrm{P}(f)} \{x\} \times \Omega(F_0^{p_x}|_{I_x})$ where p_x denotes the period of x. By the construction of F_0, x_0 is isolated in $\mathrm{P}(f)$ and $F_0|_{I_{x_0}} = g$, i.e., $\Omega(F_0|_{I_{x_0}}) = \Omega(g)$. Consequently, (5) implies that $z_0 \notin \overline{\bigcup_{x \in \mathrm{P}(f)} \{x\} \times \Omega(F_0^{p_x}|_{I_x})}$ since the set of nonwandering points is always closed.

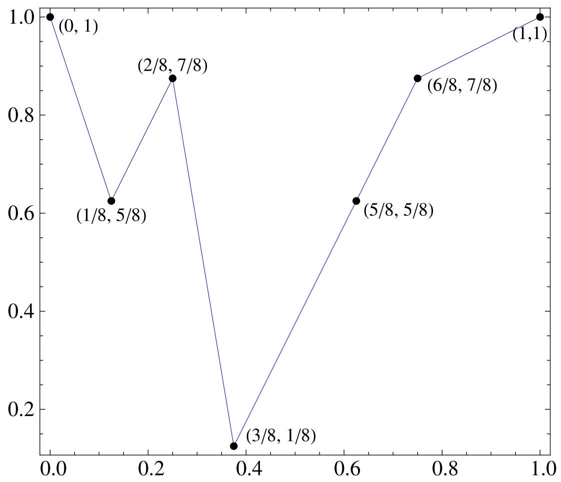

Fig. 1. The piecewise linear map g.

It remains to show that $z_0 = (x_0, y_0) \in \Omega(F_0)$. Let $\mathcal{W}_0 \subseteq \mathbb{I}^2$ be an open neighborhood of z_0. Clearly, we may assume that $\mathcal{W}_0 = \mathcal{U}_0 \times \mathcal{V}_0$ where $\mathcal{U}_0, \mathcal{V}_0 \subseteq \mathbb{I}$ and $\mathcal{U}_0 \subseteq U$. We are going to show that there exists $k \in \mathbb{N}$ for which $F^k(\mathcal{W}_0) \cap \mathcal{W}_0 \neq \emptyset$.

Since x_0 is an attracting fixed point there exists $x \in \mathcal{U}_0 \setminus \{x_0\}$ for which $f^n(x) \in \mathcal{U}_0$ for each $n \in \mathbb{N}$. Consequently, it is sufficient to show that, for some $k \in \mathbb{N}$,

$$y_0 \in \mathcal{V}_k, \quad \text{where } \mathcal{V}_i = \pi_2\big(F^i(\mathcal{W}_0 \cap I_x)\big),\ i = 1, 2, \ldots \tag{7}$$

and π_2 is the projection on the second coordinate. By the construction of F, $\mathcal{V}_i = g^{i-1}(\mathcal{V}_1)$ for any $i = 1, 2, \ldots$. Finally, since $\mathcal{V}_1$ contains some left-side open neighborhood $\mathcal{W}$ of a, (6) proves (7), i.e., $z_0 \in \Omega(F_0)$. This finishes the proof. □

5. Concluding remarks and an open problem

The results contained in Theorems A and B help to understand the structure of the set of nonwandering points of the two-dimensional discrete dynamical systems generated by maps on $\mathcal{C}^*_\Delta(\mathbb{I}^2)$. The dynamics of a system on its set of nonwandering points is very important as we show through the next definition used for introducing the open problem.

Definition 3. It is said that the *center* $C(\psi)$ of a dynamical system $(\mathbb{X}, \psi)$ is the closure of *Rec*(ψ).

Another two equivalent definitions of center are the following (see Proposition 4 proved in [2]): it is well known that, for a dynamical system $(\mathbb{X}, \psi)$ where ψ is a continuous map, the set $\Omega(\psi)$ is nonempty, closed, ψ-invariant and, in general, $\Omega(\psi|_{\Omega(\psi)}) \subseteq \Omega(\psi)$. Consequently, if we put $\Omega_1 = \Omega(\psi)$ and $\Omega_{n+1} = \Omega(\psi|_{\Omega_n})$ for any nonlimit ordinal number n, we can obtain a decreasing sequence of nonempty, closed and ψ-invariant sets. Further, for any limit ordinal number γ we denote

$$\Omega_\gamma = \bigcap_{n<\gamma} \Omega_n.$$

In general, we obtain a transfinite decreasing sequence $\{\Omega_n\}$ of nonempty, closed and ψ-invariant sets. Accordingly to the Baire–Hausdorff theorem, we have $\Omega_\beta = \Omega_{\beta+1} = \cdots$ for some finite or countable ordinal number β. Then $C_\Omega(\psi) = \Omega_\beta$ is the *center* of the map ψ and $d_\Omega(\psi) = \beta$ is called the *depth of the center* (see [6]).

The third definition of the centre is very similar to the previous one. The only change is that we use the closure of the set $\omega(\varphi)$ instead of $\Omega(\psi)$. Recall that $\omega(\psi)$ denotes the union of the ω-limit sets of all points in $\mathbb{X}$. It is known that $\overline{\omega(\psi)}$ is a nonempty, closed and ψ-invariant set for which $\overline{\omega(\psi|_{\overline{\omega(\psi)}})} \subseteq \overline{\omega(\psi)}$. Thus, let $\omega_1 = \overline{\omega(\psi)}$ and let, $\omega_{n+1} = \overline{\omega(\psi|_{\omega_n})}$ for any nonlimit ordinal number n. If ϑ is a limit ordinal number then we put

$$\omega_\vartheta = \bigcap_{n<\vartheta} \omega_n.$$

Again, in general, we obtain a transfinite decreasing sequence $\{\omega_n\}$ of nonempty, closed and ψ-invariant sets. Again by the Baire–Hausdorff theorem, we have $\omega_\alpha = \omega_{\alpha+1} = \cdots$ for some finite or countable ordinal number α. Then $C_\omega(\psi) = \omega_\alpha$ is the *center* of the map ψ and $d_\omega(\psi) = \alpha$ is called the *depth of the center*.

Proposition 4 *(Akin). Let $\psi : \mathbb{X} \to \mathbb{X}$ be a continuous map. Then $C(\psi) = C_\omega(\psi) = C_\Omega(\psi)$.*

In Efremova [12] was stated that the depth of the center of elements on $\mathcal{C}^*_\Delta(\mathbb{I}^2)$ is at most 2. We conjecture as an open problem which is possible to construct an element on $\mathcal{C}^*_\Delta(\mathbb{I}^2)$ having depth of the center equal to 3. Our feeling is that this can be done using the new information given in Theorems A and B.

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